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Normal forms of "near similarity" transformations and linear matrix equations

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Abstract

A formal solution to a linear matrix differential equation with irregular singularity $t^{1-r}Y'(t) = A(t)Y(t)$, where $r \in \mathbb{Z}^+$ and the matrix-valued function $A(t)$ is analytic at $t = \infty$, was obtained via reduction of the coefficient $A(t)$ to its Jordan form. The same approach was also utilized to find formal solutions to difference equations and to singularly perturbed differential equations. The linear change of variables $Y = TX$, where X is the new unknown matrix, generates the transformation $A \longrightarrow T^{-1}AT - t^{1-r}T^{-1}T'$. When $r > 0$, this transformation can be considered as a "small perturbation" of the similarity transformation $A \longrightarrow T^{-1}AT$. Various normal forms of these two transformations could be found in the literature. The emphasis of the present paper is to describe some classes of "near similarity" transformations that have the same normal forms as $A \longrightarrow T^{-1}AT$. Obtained results are used to construct formal solutions to matrix functional equations and to discretized differential equations. © 2000 Elsevier Science Inc. All rights reserved.

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0. Introduction

0.1. Normal forms of the similarity transformation

Consider an $n \times n$ complex matrix-valued function

$$A(t) = \sum_{k=0}^{\infty} A_k t^{-k}, \tag{0.1}$$

where $n \in \mathbb{N}$ and the series has a nonzero radius of convergence. Convergence or divergence of the series (0.1) is not essential for the purposes of this paper, so we will talk about the matrix given by the *formal holomorphic series* (0.1). The coefficient matrices A_k, in general, could be matrices over any algebraically closed field $\mathscr{F}$ of characteristic zero, $t \notin \mathscr{F}$. By a *similarity transformation* we mean the transformation

$$A(t) \longrightarrow T^{-1}(t)A(t)T(t), \tag{0.2}$$

where the matrix

$$T(t) = \sum_{k=-m}^{\infty} T_k t^{-k/p} \tag{0.3}$$

with some $p \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ and T_k matrices over $\mathscr{F}$. (Here and henceforth we use $\mathbb{Z}^+$ and $\mathbb{N}$ to distinguish between positive integer and nonnegative integer numbers.) It is assumed that $\det T(t) \not\equiv 0$, i.e. that the inverse matrix $T^{-1}(t)$ exists and has the form (0.3) with, possibly, different m.

Suppose $p = 1$. A transformation (0.2) is called *holomorphic* if both T and T^{-1} have representation (0.3) with $m = 0$. Otherwise, the transformation is called *meromorphic*. Clearly, (0.2) is holomorphic if and only if $m = 0$ and the matrix T_0 in (0.3) is invertible. If $p > 1$, the corresponding transformations are called *root holomorphic* and *root meromorphic*, respectively. The corresponding terminology is used for the matrix series $T(t)$ in (0.2).

Normal forms of matrices (0.1) with respect to the above mentioned types of similarity transformations are known. For example, these normal forms, as well as normal forms of singular differential operators, were discussed in [8,12]. The results are summarized in Table 1 (explanations are given below). Related bibliography can be found in [8] (see also [15,16]).

Table 1

$p\backslash m$	Holomorphic	Meromorphic
Integer	A-form	F-form
Root	T-form	J-form

Consider $A(t)$ as a matrix over the ring $\mathscr{F}[[t^{-1}]]$ of (scalar) formal holomorphic series with coefficients in $\mathscr{F}$. The quotient field of this ring is the field $\mathscr{F}((t^{-1}))$ of formal meromorphic series. Therefore, the F-form from Table 1 is just the standard first normal formal of a matrix (we call it Frobenius form), i.e. a direct sum of the blocks

$$\lambda I + H + t^{-1}\tilde{A}(t). \tag{0.4}$$

Here $\lambda \in \mathscr{F}$, I is the identity matrix,

$$H = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

is the upper shift matrix and $\tilde{A}$ is a formal holomorphic series with all nonzero elements located at the first column.

The field of all formal root-meromorphic series $\mathscr{F}\{t^{-1}\}$ is the algebraic closure $\mathscr{F}((t^{-1}))$. Therefore, the J-form from Table 1 is the standard Jordan form. The T-form (triangular form) of a matrix $A(t)$ is an upper triangular matrix of formal root-holomorphic series. It can be obtained from the J-form by means of Birkhoff's factorization lemma (Lemma 3.1). Finally, the A-form from Table 1 (Arnold form) is a direct sum of the blocks

$$\lambda I + N + t^{-1}\tilde{A}(t), \tag{0.5}$$

where $\lambda \in \mathscr{F}$; the nilpotent matrix N is a direct sum of upper shift matrices H_i and $\tilde{A}$ is a matrix of formal holomorphic series with the following special structure: all nonzero elements of $\tilde{A}(t)$ are located either in the first column or in the last row (whichever has less entries) of each block in the block structure, induced on $\tilde{A}(t)$ by H_i (see [1,16] or [8] for more details).

0.2. Near similarity transformations

Let us consider the orbit of the matrix $A(t)$ with respect to holomorphic transformations. This orbit consists of all matrices $B(t)$, such that $B = T^{-1}AT$, where T is a holomorphic transformation (0.2). Rewriting the latter equation as

$$T(t)B(t) = A(t)T(t), \tag{0.6}$$

we can easily obtain the recurrent system of equations for coefficients of $B(t)$, $T(t)$. Namely, taking $T_0 = I$, we get $B_0 = A_0$ and

$$[A_0, T_m] = B_m - A_m + C_m, m \geqslant 1, \tag{0.7}$$

where $[A_0, \cdot]$ denotes the commutator and C_m is a matrix, known at the step m.

Without loss of generality, we can assume that the matrix A_0 is in the Jordan form. In order to solve (0.7) for B_m and T_m on each step m recurrently, we require B_m to be in the complement of the image of commutator $[A_0, \cdot]$, i.e. $B_m \in \mathrm{CoIm}[A_0, \cdot]$. For example, if A_0 has n distinct eigenvalues we can choose $\mathrm{CoIm}[A_0, \cdot]$ to consist of all diagonal matrices. Thus, the holomorphic transformation (0.2) with $T(t)$ defined by (0.7) diagonalizes the matrix $A(t)$, i.e. reduces $A(t)$ to its J-form. In the general case the system (0.7) allows us to reduce $A(t)$ to its A-form, described above.

Roughly speaking, by a *near similarity transformation* we understand a perturbation of the transformation (0.2) that preserves the structure of the recurrent system (0.7).

0.3. Some related equations

Consider a matrix linear differential equation

$$t^{1-r}Y'(t) = A(t)Y(t), \tag{0.8}$$

where $\mathscr{F} = \mathbb{C}$, $A(t)$ is given by (0.1) and r is a natural (or, more generally, a rational) number, called the Poincare rank of (0.8). This equation has an irregular singularity at infinity if $r > 0$.

The main obstacle in solving the matrix equation (0.8) is that the values of $A(t)$ do not commute for different t (or coefficients A_k do not commute with each other if the series (0.1) is formal). The standard way to proceed with (0.8) is to try to split it into a direct sum of commutative blocks by means of linear changes of variables $Y = TX$. The new equation for $X(t)$ becomes

$$t^{1-r}X'(t) = [T^{-1}(t)A(t)T(t) - t^{1-r}T^{-1}(t)T'(t)]X(t).$$

Thus, the change $Y = TX$ generates the transformation

$$A(t) \longrightarrow T^{-1}(t)A(t)T(t) - t^{1-r}T^{-1}(t)T'(t) \tag{0.9}$$

of the coefficient of the equation. It is easy to see that (0.9) is a near similarity transformation if $r > 0$.

It turns out that the transformations (0.9) and (0.2) reduce $A(t)$ to essentially the same normal forms (given in Table 1). In particular, $A(t)$ can be reduced to its J-form $J(t)$ by a root-meromorphic transformation (0.9) with some $T(t)$. As any J-form is commutative, we can easily obtain a solution $X(t) = \exp(\int^t u^{r-1}J(u)\,\mathrm{d}u)$ to the equation $t^{1-r}X'(t) = J(t)X(t)$ and thus get the Hukuhara–Turrittin formal solution

$$Y(t) = T(t)t^N \mathrm{e}^{Q(t)} \tag{0.10}$$

to (0.8). Here N is a nilpotent matrix in Jordan form and $t\frac{\mathrm{d}}{\mathrm{d}t}Q(t)$ is a diagonal matrix, polynomial in $t^{1/p}$ for some $p \in \mathbb{Z}^+$ of degree not exceeding pr; moreover, the commutator $[N, Q(t)] = 0$ (see, for example, [15] for more details).

The same approach was utilized in [14] to obtain a formal solution (3.4) to the matrix linear difference equation

$$Y(t+1) = A(t)Y(t), \tag{0.11}$$

where the matrix $A(t)$ is invertible. It can be easily verified that the difference equation (0.11), as well as the more general functional equation

$$Y(\phi(t)) = A(t)Y(t), \tag{0.12}$$

where the scalar function $\phi(t) - t$ is holomorphic at $t = \infty$ (or, more generally, $\phi(t) - t$ is a root-meromorphic series in t^{-1} of order $o(t)$), generate near similarity transformations

$$A(t) \longrightarrow T^{-1}(t+1)A(t)T(t) \tag{0.13}$$

and

$$A(t) \longrightarrow T^{-1}(\phi(t))A(t)T(t), \tag{0.14}$$

respectively. Normal forms of linear operators are closely related with invariants and classification of these operators. For singular differential operators defined by (0.8), this scope of problems were studied in [2,9]. The formal classification of difference operators defined by (0.11) was obtained in [10]. It was also observed there that functional equations (0.12) can be treated similarly.

Consider now a singularly perturbed matrix differential equation

$$\varepsilon^r \partial_x Y(x,\varepsilon) = A(x,\varepsilon)Y(x,\varepsilon), \tag{0.15}$$

where ε is a small parameter, $r \in \mathbb{Z}^+$, $x \in \mathbb{C}$ and ∂_x denotes differentiation with respect to x. We assume $A(x,\varepsilon) = \sum_{k=0}^{\infty} A_k(x)\varepsilon^k$ to be a formal holomorphic series in ε, where the coefficients $A_k(x)$ are matrices over the field $\mathscr{F}$ of germs of root-meromorphic functions (in x^{-1}) at $x = \infty$. (More generally, we can consider $\mathscr{F} = \mathbb{C}\{x^{-1}\}$ to be the field of formal root-meromorphic series in x^{-1}.) Strong interest in Eq. (0.15) is motivated by the turning point problems (see, for example, [16]).

The field $\mathscr{F}$ is an algebraically closed differential field with the differentiation ∂_x. Denoting $\varepsilon = t^{-1}$, we see that Eq. (0.15) generates the near similarity transformation

$$A \longrightarrow T^{-1}AT - t^{-r}T^{-1}\partial_x T. \tag{0.16}$$

It was shown ([11,8, Section 2.5]) that normal forms of this transformation are the same as in Table 1 and that a formal solution to (0.15) is given by $Y(x,\varepsilon) = T(x,\varepsilon)\exp(\varepsilon^{-r}\int^x J(u,\varepsilon)\,\mathrm{d}u)$, where $J(x,\varepsilon)$ is the J-form of $A(x,\varepsilon)$ and $T(x,\varepsilon)$ is the corresponding transformation.

Another example of a near similarity transformation is provided by the difference equation

$$Y(x+\varepsilon,\varepsilon) = A(x,\varepsilon)Y(x,\varepsilon), \tag{0.17}$$

where ε is a small parameter and the matrix $A(x,\varepsilon)$ was specified in the previous example. This equation naturally appears in the theory of finite difference approximations of (0.8). Denoting $\varepsilon = t^{-1}$, we get the corresponding near similarity transformation as

$$A(x,t^{-1}) \longrightarrow T^{-1}(x+t^{-1},t^{-1})A(x,t^{-1})T(x,t^{-1}). \tag{0.18}$$

0.4. Brief outline of the paper

Examples from Section 0.3 provide a strong motivation to study near similarity transformations. Normal forms of these transformations is the main subject of the paper. In Section 1, we define a fairly large class of near similarity transformations, compatible with the “intuitive” understanding of Section 0.2. We also show that a matrix $A(t)$ over $\mathscr{F}[[t^{-1}]]$ can be reduced to its A-forms by a holomorphic near similarity transformation. In order to reduce $A(t)$ to F, J and T-forms successively (Sections 2 and 3), we have to impose the successive Assumptions F, J and T, thus narrowing the corresponding classes of transformations. However, even the most narrow class still contains all the near similarity transformations from Section 0.3 with $\mathscr{F}=\mathbb{C}$. In Section 3.3, we utilize the J-form of (0.14) to derive a formal solution (3.6) to the functional equation (0.12). This solution seems to be new (though results of [10] are related to the reduction to J-form). In fact, it is very similar to the formal solution (3.4) of the difference equation (0.11). The main difference between these solutions is that the Γ-function in (3.4) has to be replaced by some “special function” $g(t)$, satisfying $g(\phi(t)) = tg(t)$. The latter equation was considered in Section 3.4. Here we found a simple condition that allows us to reduce the functional equation (0.12) with a root-meromorphic coefficient $A(t)$ to the difference equation (0.11), which also has a root-meromorphic coefficient. Some particular examples of $\phi(t)$ were considered.

The last two examples of near similarity transformations from Section 0.3, i.e. transformations (0.16) and (0.18), do not satisfy Assumption J. These two examples are considered separately in Section 4. In particular, in Sections 4.1 and 4.3 we reproduce the approach of [11] to construct a formal matrix solution to (0.15) and to reduce $A(x,\varepsilon)$ to its J-form by the transformation (0.16). In Sections 4.4 and 4.5, this approach was adjusted to obtain first a formal matrix solution (4.24) to (0.17), and then to reduce $A(x,\varepsilon)$ to its J-form by the transformation (0.18). The formal solution (4.24) seems to be a new result for this important class of difference equations.

1. Normal forms

1.1. Definition of near similarity transformations

Examples from the previous section show that some matrix equations generate near similarity transformations. Let us consider a general matrix equation of the form

$$L[Y(t)] = A(t)Y(t), \tag{1.1}$$

where the linear operator L is acting on matrix-valued functions (or on formal matrix series) over the field $\mathscr{F}$ and satisfies

$$L[TX] = F[T]L[X] + G[T]X$$

with F, G being also linear operators. (In the case $\mathbb{C}$ is a proper subfield of $\mathscr{F}$ we assume that the operators L, F and G are linear with respect to $\mathbb{C}$ and additive with respect to $\mathscr{F}$.) Eq. (1.1) generates the transformation

$$\begin{aligned} A &\longrightarrow (F[T])^{-1}AT - (F[T])^{-1}G[T] \\ &= T^{-1}AT - \Big((F[T])^{-1}G[T] + \{(F[T])^{-1} - T^{-1}\}AT\Big). \end{aligned} \tag{1.2}$$

For example, if L is a differential operator $L = t^{1-r}\mathrm{d}/\mathrm{d}t$ or $L = t^{-m}\partial_x$ (as in (0.8) and (0.15)) then $F = \mathrm{id}$, $G = L$, where id denotes the identity operator. The case $L[Y(t)] = Y(\phi(t))$ (Eq. (0.12)) yields $F = L$, $G = 0$. If $\Delta Y(t) = Y(t+1) - Y(t)$, then $L = \varDelta$ yields $F[T(t)] = T(t+1)$, $G = L$. Direct computations show that the transformation (0.18) yields $G = 0$ and

$$F = \mathrm{id} + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\partial_x}{t}\right)^k = \exp\left(\frac{\partial_x}{t}\right). \tag{1.3}$$

In order to define near similarity transformations we need to specify action of linear operators F, G on formal root-meromorphic series $T(t)$. Let Q denote a matrix over the field $\mathscr{F}$ and $k \in \mathbb{Z}$, $p \in \mathbb{Z}^+$.

Definition 1.1. A transformation (1.2) is called a near similarity transformation if for all Q, k, p: (1) $F[Qt^{-k/p}], G[Qt^{-k/p}]$ are matrix formal power series in $t^{-1/p}$, and (2) there exists some positive $r \in \mathbb{Q}$ such that both $(F - id)[Qt^{-k/p}]$ and $G[Qt^{-k/p}]$ are of the order $\mathrm{o}(t^{-k/p-r})$ as $t \to \infty$.

It is easy to check that conditions (1) and (2) are satisfied for all the examples mentioned above. Thus, the corresponding transformations are near similarity transformations. Let $B(t)$ denote the right-hand side of (1.2). Then $F[T]B = AT - G[T]$. Conditions (1) and (2) from the definition ensure that the system of recurrent equations for the coefficients of $T(t)$ and of $B(t)$ has the same form as (0.7), so that Definition 1.1 is consistent with the "intuitive understanding", introduced in Section 0.2. As the reduction to the A-form is based solely on the structure of the system (0.7) (see [1,16] or [8] for details of the proof), we get the following statement:

Statement 1.1. *The matrix $A(t)$ is reducible to A-form by means of a holomorphic near similarity transformation* (1.2).

1.2. F-form

We can assume now that the matrix $A(t)$ is already reduced to A-form so that

$$A_0 = \lambda I + \mathrm{diag}(H_1, \ldots, H_l)\,, \tag{1.4}$$

where $\lambda \in \mathscr{F}, l \in \mathbb{Z}^+$ and H_j are upper shift matrices. Let $\tilde{A}_{ij}$ denote (i, j)th block of the matrix $\tilde{A} = t[A(t) - A_0]$ in the block structure induced by A_0. Suppose that among the "lower triangular" blocks $\tilde{A}_{ij}$, where $i > j$, there exists at least one nonzero block. Then we can find some natural α and $k < n$ such that $B_0 - A_0$, where B_0 is the leading term of $B = T^{-1}AT$ with

$$T(t) = \mathrm{diag}\big(I_k, t^{-\alpha} I_{n-k}\big) \tag{1.5}$$

is a nonzero strictly lower triangular matrix. (Here I_l denotes the l-dimensional identity matrix.)

Next, the matrix $B(t)$ can be reduced to its A-form and the whole process can be repeated. Using one of the Turrittin's lemmas (see [15, Lemma 19.4]), one can show [8,13] that after a finite number of such steps either $A(t)$ becomes block diagonal or $A_0 = \lambda I + H$, where H is the n-dimensional upper shift matrix. Thus, $A(t)$ is reduced to its F-form.

The matrix (1.5) is called a shearing matrix, and the transformation (1.2) with a shearing matrix T is called a shearing transformation. The general form of a shearing matrix is

$$S(t) = \mathrm{diag}\big(t^{\alpha_1}, \ldots, t^{\alpha_n}\big)\,, \tag{1.6}$$

where $\alpha_j,\ j = 1, 2, \ldots, n$ are rational numbers.

In order to generalize the reduction to F-form for near similarity transformations, we need to impose an additional requirement on the operators F and G.

Assumption F. The operators F, G act entry-wise on diagonal matrices, i.e. if all the entries of the matrix Q from Definition 1.1 but the entry $(j, j),\ j \in \{1, 2, \ldots, n\}$, are zeroes, then so are the entries of $F[Qt^{-k/p}], G[Qt^{-k/p}]$ for all $k \in \mathbb{Z},\ p \in \mathbb{Z}^+$.

Assumption F can be illustrated, for example, by the operator $G[T(t)] = \frac{\mathrm{d}}{\mathrm{d}t} T^{\mathrm{T}}(t)$, where A^{T} denote the matrix transponent to A. This operator satisfies Assumption F but does not act entry-wise.

An immediate consequence of Assumption F is that for any shearing matrix S both $S^{-1}(F - id)[S]$ and $S^{-1}G[S]$ are of the order $\mathrm{O}(t^{-r})$, where r is the same as in Definition 1.1.

Statement 1.2. *The matrix $A(t)$ is reducible to F-form by means of a meromorphic near similarity transformation* (1.2), *satisfying Assumption F.*

Indeed, according to Assumption F, $F[S] = S(I + \mathrm{O}(t^{-r}))$, so that $(F[S])^{-1} = (I + \mathrm{O}(t^{-r}))^{-1}S^{-1}$. Then the leading terms of $S^{-1}AS$ and $(F[S])^{-1}AS - (F[S])^{-1}G[S]$, where the shearing matrix S is defined by (1.5), coincide. We can reduce $B = (F[S])^{-1}AS - (F[S])^{-1}G[S]$ to its A-form and continue the process.

By virtue of the same arguments, as above, the matrix $A(t)$ could be reduced to its F-form.

1.3. J-form

As a first step towards the reduction to J-form we show that a root-meromorphic near similarity transformation satisfying Assumption F can reduce $A(t)$ to a diagonal matrix up to the order $\mathrm{O}(t^{-r})$. More precisely:

Statement 1.3. *The matrix $A(t)$ is reducible to a direct sum of the blocks*

$$\lambda(t)I + Dt^{-r} + \mathrm{o}(t^{-r}), \tag{1.7}$$

where $\lambda(t)$ is a (scalar) polynomial in $t^{-1/p}$ over $\mathscr{F}$ with some $p \in \mathbb{Z}^+$ of order not greater than $pr - 1$ and D is a matrix over $\mathscr{F}$.

Proof. According to Statement 1.2, we can assume that $A(t)$ has the form

$$A(t) = \lambda_0 I + H + t^{-1}\tilde{A}(t), \tag{1.8}$$

where only the first column $\mathrm{Col}\,(\tilde{a}_1(t), \tilde{a}_2(t), \ldots, \tilde{a}_n(t))$ may contain nonzero elements. Let us apply the shearing transformation (1.2)–(1.8) with the shearing matrix

$$S(t) = \mathrm{diag}\Big(1, t^{-\alpha}, t^{-2\alpha}, \ldots, t^{-(n-1)\alpha}\Big), \tag{1.9}$$

where

$$\alpha = \min_{1\leqslant k\leqslant n} \frac{\deg \tilde{a}_k(t)}{k}. \tag{1.10}$$

Here deg of formal power series $a(t)$ (in powers of $t^{-1/p}$) is defined as the exponent of the leading term, taken with the opposite sign. For example, $\deg t^{-k} = k$. If $a(t) \equiv 0$, then $\deg a(t) = \infty$.

If $\tilde{A}(t) \neq 0$, then (1.10) defines some finite α. The result of the transformation is the matrix

$$\begin{aligned} B(t) =& \lambda_0 I + (F[S])^{-1}(H + t^{-1}\tilde{A})S \\ & -(F[S])^{-1}\left[(\lambda_0(F - \mathrm{id})[S] + G[S]\right). \end{aligned} \tag{1.11}$$

The last (third) term in (1.11) is of degree not less than r. Then, according to (1.9) and (1.10), $\deg B_{i,i+1}(t) = \alpha$, while $\deg(B - \lambda_0 I)_{1,j}(t) \geqslant \alpha$ and at least one of the latter inequalities is nonstrict. Here $i = 1, \ldots, n-1,\ j = 1, \ldots, n$ and $A_{k,l}$ denotes the (k,l)th entry of the matrix A. Thus

$$B(t) = \lambda_0 I + t^{-\tilde{\alpha}}\tilde{B}(t), \tag{1.12}$$

where $\tilde{\alpha} = \min\{\alpha, r\}$.

Note that reduction to (1.7) is already completed if $\alpha \geqslant r$. Otherwise, there are two options: (1) the leading term $\tilde{B}_0$ of $\tilde{B}(t)$ has at least two distinct eigenvalues; (2)

$\tilde{B}_0$ has a single eigenvalue $\tilde{\lambda}$ (note that $\tilde{\lambda} \neq 0$), which implies that the first column of $\tilde{B}_0$ consists of nonzero entries and thus $\alpha \in \mathbb{Z}^+$. In the first case, we can split the matrix $B(t)$ into a direct sum of diagonal blocks (note that α can be rational) by reducing $\tilde{B}(t)$ to its A-form. Note that the fact that $B(t) = \lambda_0 I + t^{-\alpha}\tilde{B}(t)$ will not affect the form of the corresponding recurrent system (0.7) for $\tilde{B}(t)$ since $\deg \lambda_0 F^{-1}[T]T \geqslant r$ for any holomorphic transformation T. In fact, the latter inequality holds for any shearing transformation T as well. In the second case $\tilde{B}_0 = \tilde{\lambda} I + H$, so that we can reduce $\tilde{B}(t)$ in (1.12) to its F-form. Then we apply to $\tilde{B}(t)$ the shearing transformation with the corresponding matrix (1.9) and so continue the process. Thus, after a final number of steps we will reduce $A(t)$ to a direct sum of blocks, where each block has form (1.7) (note that any scalar block has the form (1.7)). □

Remark 1.1. In general, Statement 1.3 does not provide with a reduction to J-form. However, for the similarity transformation (0.2) this reduction can be easily completed. Observe that in this case we can choose as large r as we want. For the matrix (1.7) we consider two cases: (1) it has a single eigenvalue $\lambda(t)$, (2) it has at least two different eigenvalues $\lambda(t)$ and $\mu(t)$. In the first case it can be reduced to $\lambda(t)I + H$ by a special holomorphic transformation (see [11], Theorem 1.2). In the second case we can choose $r > \deg(\lambda(t) - \mu(t))$ and split the matrix.

2. Reduction to J-form

2.1. Formulation of the theorem

For near similarity transformations we need to impose more assumptions on the operators F, G in order to bring the matrix $A(t)$ to J-form. Here and henceforth we assume $\mathscr{F} = \mathbb{C}$.

Assumption J. The field $\mathscr{F} = \mathbb{C}$; For every matrix Q over $\mathbb{C}$ and for every $q \in \mathbb{Q}$

$$(F - id)[Qt^q] = Qt^{q-r} f_q(t) \quad \text{and} \quad G[Qt^q] = Qt^{q-r} g_q(t), \tag{2.1}$$

where f_q, g_q are scalar functions, holomorphic at infinity (or having formal power series expansion in $t^{-1/p}$ for some $p \in \mathbb{Z}^+$). We assume that either $\deg g_q = 0$ and $\deg f_q > 0$ for all $q \in \mathbb{Q}\backslash\{0\}$ or $\deg f_q = 0$ and there exists a root-holomorphic formal power series $\kappa(t)$ such that

$$\kappa(t) f_q(t) + g_q(t) = 0 \tag{2.2}$$

for all $q \in \mathbb{Q}\backslash\{0\}$. We also assume

$$f_0(t) \equiv g_0(t) \equiv 0, \quad \text{i.e.} F[Q] = Q \ \text{ and } \ G[Q] = 0 \tag{2.3}$$

for every constant matrix Q.

By analogy with differential operators (0.8), we call r the Poincare rank of (1.2). Note that condition (2.1) implies that actions of the operators F and G are entry-wise.

In the case of transformation (0.9) we have $f_q \equiv 0$ and $g_q = q$ for all $q \in \mathbb{Q}$. For the transformation (0.13), $f_q(t) = (1 + t^{-1})^q - 1$ and $g_q \equiv 0$, so that $r = 1$. Finally, $f_q(t) = (\phi(t)/t)^q - 1$ and $g_q \equiv 0$ for the transformation (0.14). In this case $r = \deg(\phi(t) - t)$. The last two examples from Section 0.3 (transformations (0.16) and (0.18)) do not satisfy Assumption J since in these cases $\mathscr{F} \neq \mathbb{C}$. Therefore these two important examples will be considered separately in Section 4.

Theorem 2.1. *The matrix $A(t)$ is reducible to J-form by means of a root-meromorphic near similarity transformation* (1.2), *satisfying Assumption J.*

Proof. Without any loss of generality we can assume that $A(t)$ is in the form (1.7)

$$A(t) = \sum_{j=0}^{m-1} \lambda_j t^{-j/p} I + Dt^{-r} + \sum_{j=1}^{\infty} A_{m+j} t^{-(m+j)/p}, \tag{2.4}$$

where $\lambda(t) = \sum_{j=0}^{m-1} \lambda_j t^{-j/p},\ r = m/p$ and D is in the Jordan form (the latter can be accomplished due to the assumptions $(F - id)[Q] = G[Q] = 0$ for any constant matrix Q). We reduce $A(t)$ to

$$B(t) = \sum_{j=0}^{m-1} \lambda_j t^{-j/p} I + Dt^{-r} + \sum_{j=1}^{\infty} B_{m+j} t^{-(m+j)/p}, \tag{2.5}$$

by means of the transformation (1.2), where

$$T(t) = I + \sum_{j=1}^{\infty} T_j t^{-j/p}. \tag{2.6}$$

Note that, according to Assumption J, we can expand

$$F[T] = \sum_{k=0}^{\infty} F_k t^{-k/p}, \qquad G[T] = \sum_{k=0}^{\infty} G_k t^{-k/p}, \tag{2.7}$$

where $F_k = T_k,\ G_k = 0$ when $k \leqslant m$ and

$$F_{m+j} = T_{m+j} + \sum_{l=0}^{j-1} f_{j-l/p,l} T_{j-l}, \qquad G_{m+j} = \sum_{l=0}^{j-1} g_{j-l/p,l} T_{j-l} \tag{2.8}$$

when $k = m + j,\ j > 0$. Here $f_{q,l}$ and $g_{q,l}$, where $q \in \mathbb{Q}$, denote lth coefficients in the expansions of the functions f_q and g_q, respectively, in powers of $t^{-1/p}$.

Comparing the coefficients of $t^{-(m+j)/p}$, $j = 1, 2, \ldots$ of the equation

$$F[T]B = AT - G[T], \tag{2.9}$$

we get the recurrent system

$$B_{m+j} + T_1 B_{m+j-1} + \cdots + T_m B_j + F_{m+1} B_{j-1} + \cdots + F_{m+j} \lambda_0$$

$$
\begin{aligned}
&= A_{m+j} + A_{m+j-1}T_1 + \cdots + A_j T_m + A_{j-1}T_{m+1} \\
&\quad + \cdots + \lambda_0 T_{m+j} - G_{m+j},
\end{aligned}
\tag{2.10}
$$

where $A_k = B_k = \lambda_k I$ for $k = 0, 1, \ldots, m-1$ and $A_m = B_m = D$.

We want to solve (2.9) for T_j and B_{m+j} at every step j. Taking into account (2.4)–(2.8), we can rewrite (2.10) as

$$
\begin{aligned}
&[T_1, D] + \mu_1 T_1 = A_{m+1} - B_{m+1}, \quad j = 1, \\
&[T_j, D] + \mu_j T_j = C_{m+j} - B_{m+j}, \quad j > 1,
\end{aligned}
\tag{2.11}
$$

where

$$\mu_j = f_{j/p,0}\lambda_0 + g_{j/p,0} \tag{2.12}$$

and the matrix C_{m+j} is known at the step j. Consider the two following cases.

2.2. *Case 1*

Suppose that in Assumption J $\deg g_q = 0$ and $\deg f_q > 0$ for all $q \in \mathbb{Q}\backslash\{0\}$. Then $\mu_j \neq 0$ for all $j \in \mathbb{Z}\backslash\{0\}$ and system (2.11) can be solved uniquely for T_j with all $B_{m+j} = 0$, provided that no eigenvalues d_1, d_2 of the matrix D satisfy $d_1 - d_2 = \mu_j$ for any $j = 1, 2, \ldots$ Thus, the matrix $A(t)$ is reduced to

$$B(t) = \lambda(t)I + t^{-r}D, \tag{2.13}$$

where D is in the Jordan form. This is J-form of $A(t)$.

Remark 2.1. If $A(t) = \lambda(t)I + \mathrm{o}(t^{-r})$, then it can be reduced to its J-form $\lambda(t)I$ by a holomorphic in $t^{-1/p}$ transformation.

Suppose now that k is the first natural number, such that $d_2 - d_1 = \mu_k$. Since we can solve the first $k-1$ equations (2.11) as described above, we can assume $A_{m+j} = 0$ for $j = 1, \ldots, k-1$. Let

$$D = \mathrm{diag}(D_1, D_2), \tag{2.14}$$

where D_1 is the direct sum of all Jordan blocks of D with the eigenvalue d_1. Thus, d_2 is an eigenvalue of D_2. We will show that the shearing transformation with

$$S(t) = \mathrm{diag}(I, t^{-k/p}I), \tag{2.15}$$

where the diagonal structure of S is induced by (2.14), preserves the first $m-1$ terms in (2.4) and changes D to $D - \mathrm{diag}(0, \mu_k I)$. So, the number of pairs of eigenvalues with nonzero differences will decrease at least by 1. As the maximal number of such pairs is $n(n-1)/2$ (this is the case when D has distinct eigenvalues), after a finite number of the above mentioned transformations the matrix D will have no eigenvalues with the difference μ_k, where $k = 1, 2 \ldots$

Indeed, the shearing transformation with the matrix (2.15) reduces $A(t)$ to $B(t)$, where

$$B = (F[S])^{-1}AS - (F[S])^{-1}G[S]. \tag{2.16}$$

It follows from Assumption J and (2.15) that

$$\begin{aligned} F[S] &= S + \begin{pmatrix} 0 & 0 \\ 0 & t^{-(k+m)/p} f_{k/p}(t)I \end{pmatrix} \\ &= S \cdot \begin{pmatrix} 1 & 0 \\ 0 & [1 + t^{-r} f_{k/p}(t)]I \end{pmatrix}. \end{aligned} \tag{2.17}$$

Thus

$$(F[S])^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & [1 + t^{-r} f_{k/p}(t)]^{-1}I \end{pmatrix} \cdot S^{-1}, \tag{2.18}$$

so that (2.16) yields

$$B(t) = \lambda(t)I + t^{-r}\left[D + \begin{pmatrix} 0 & 0 \\ 0 & -\mu_k I \end{pmatrix} + \mathrm{o}(1)\right]. \tag{2.19}$$

The reduced matrix B has the form (2.4), where the number of pairs of different eigenvalues of the matrix D has been reduced at least by 1. We denote B by A and start the reduction process, described above, again. It is clear that after a finite number of steps the matrix D will have no pairs of eigenvalues with the difference μ_k, where $k = 1, 2 \ldots$ So, the reduction to J-form in the Case 1 is completed. In particular, reduction to J-form is proven for singular differential operators (0.8) with positive Poincare rank r.

2.3. *Case 2*

Suppose that in Assumption J $\deg f_q = 0$ and Eq. (2.2) holds for all $q \in \mathbb{Q}\backslash\{0\}$. Consider first the special case $\kappa(t) \equiv 0$, that is the transformation (1.2) becomes

$$A \longrightarrow (F[T])^{-1}AT. \tag{2.20}$$

If $\lambda_0 \neq 0$, then reduction to J-form follows from Case 1. Therefore, the interesting case is $\lambda_0 = 0$.

Assume now that in (2.4) $\lambda(t) = t^{-l/p}(\tilde{\lambda} + \mathrm{o}(1))$, where $0 < l \leqslant m - 1$ and $\tilde{\lambda} \neq 0$. Then $A(t) = t^{-l/p}\tilde{A}(t)$, where $\tilde{A}(t) = \tilde{\lambda}I + \mathrm{o}(1)$. Let the transformation (2.20) with some $T = T_0$ reduces $\tilde{A}(t)$ to its J-form. Such a reduction is possible since $\tilde{\lambda} \neq 0$ and thus the situation corresponds to case 1. Then the matrix

$$(F[T_0])^{-1}AT_0 = t^{-l/p}(F[T_0])^{-1}\tilde{A}(t)T_0$$

is also in J-form.

Assume now that $\lambda(t) \equiv 0$. Then $A(t) = t^{-r}\tilde{A}(t)$, where $\tilde{A}(t) = D + \mathrm{o}(1)$. Let us reduce $\tilde{A}(t)$ to F-form and consider one of the corresponding blocks, which we also denote by $\tilde{A}(t)$. If the leading term $\tilde{A}_0$ is not nilpotent, then we can reduce it to J-form based on Statement 1.3 and case 1. Suppose $\tilde{A}_0 = H$, where H is the upper shift matrix. If $\tilde{A}(t) - \tilde{A}_0 \equiv 0$, then $\tilde{A}(t)$ is already in J-form. Otherwise, define α for $\tilde{A}(t)$ as in (1.10) and apply the shearing transformation (1.9). It is clear that the leading term B_0 of both matrices $(F[S])^{-1}\tilde{A}S$ and $S^{-1}\tilde{A}S$ is the same and that B_0 is not a nilpotent matrix (see the end of the proof of Statement 1.3). If B_0 has a single eigenvalue we can proceed as in case 1, otherwise we can split $\tilde{A}(t)$ into a direct sum of diagonal blocks that corresponds to different eigenvalues of B_0. Thus, we can reduce $\tilde{A}(t)$ to J-form.

Consider now the general case $\kappa(t) \not\equiv 0$. Then (2.1) together with (2.2) yield

$$G[T] = -\kappa(F - id)[T], \tag{2.21}$$

so that the right-hand side of the near similarity transformation (1.2) becomes

$$\begin{aligned} A &\longrightarrow (F[T])^{-1}\{AT + \kappa(F - id)[T]\} \\ &= (F[T])^{-1}\{A - \kappa I\}T + \kappa I. \end{aligned} \tag{2.22}$$

Let $B = A - \kappa I$ and let $J_B = (F[T])^{-1}BT$ be the J-form of B with respect to near similarity transformation (2.20), which can be obtained according to the previous arguments. Then, according to (2.22),

$$J_A = J_B + \kappa I = (F[T])^{-1}(A - \kappa I)T + \kappa I \tag{2.23}$$

is the J-form of A. The proof is completed. □

3. T-form and examples

3.1. Reduction to T-form

Reduction to T-form is based on the following factorization lemma of G.D. Birkhoff (see, for example, [2]).

Lemma 3.1. *A formal meromorphic matrix series $T(t)$ can be uniquely factorized as*

$$T(t) = H(t)P(t)t^K, \tag{3.1}$$

where K is a diagonal matrix of integer numbers, $P(t)$ is a polynomial in t upper-triangular matrix such that $P(0) = I$ and diag $P(t) = I$ *and both $H(t)$ and $H^{-1}(t)$ are formal holomorphic series.*

This lemma was utilized in [13] to reduce $A(t)$ to T-form for transformations (0.2) and (0.9). To generalize these results we need to impose additional restrictions (beyond Assumption J) on the operators F and G.

Assumption T. A matrix (0.1) can be reduced to its J-form by a near similarity transformation (1.2) and for all root-meromorphic formal series $H(t)$, $R(t)$:

(1) $F[HR] = F[H]F[R]$, i.e. F is multiplicative,

(2) $G[HR] = F[H]G[R] + G[H]R.$ (3.2)

It is easy to check that all transformations from Section 0.3, except (0.16) and (0.18), satisfy Assumption T. Moreover, if (2.2) holds, then (3.2) follows from the multiplicativity of F. This can be checked directly by substituting (2.21) into (3.2). Therefore, in this case Assumption T is simply a requirement of multiplicativity of F. Moreover, the multiplicativity of F implies that F is a substitution: $F[It] = I\phi(t)$, where $\phi(t) - t$ is a root-meromorphic series of degree $r - 1$ with some $r \in \mathbb{Q}^+$.

Theorem 3.1. *The matrix $A(t)$ is reducible to its T-form by means of a root-holomorphic near similarity transformation* (1.2), *satisfying Assumption T.*

Proof. Indeed, let a root-meromorphic transformation (2.3) with some $T(t)$ reduces $A(t)$ to its J-form $J(t)$, i.e.

$$J = (F[T])^{-1}(AT - G[T]).$$

Factorizing $T(t)$ according to Lemma 3.1 (with respect to $t^{-1/p}$) and taking into account (3.2) we get

$$(F[H])^{-1}(AH - G[H]) = (F[Pt^K]J + G[Pt^K])t^{-K}P^{-1}.$$

Note that the right-hand side of (3.2) is an upper triangular matrix. Thus the transformation

$$A \longrightarrow (F[H])^{-1}(AH - G[H]) \tag{3.3}$$

reduces A to its T-form. Note that this T-form (the right-hand side of (3.3)) is a root-holomorphic matrix series. □

Although solution of the linear equations from Section 0.3 is based on reduction to the J-form, the T-form has apparent advantages in solving linear nonhomogeneous, or, more generally, nonlinear equations (see, for example, [3,5]). This advantage, in fact, is a combination of both a convenient triangular structure and a nonsingular reducing transformation $T(t)$. However, there exists an open question about equivalence of T-forms, which, in my opinion, could be linked to the problem of classification of orbits of a constant upper triangular matrix with respect to upper triangular similarity transformations (see, for example, [7]).

3.2. Difference equations

The difference equation (0.11) generates the transformation (0.13) with the corresponding $F[T(t)] = T(t + 1)$ and $G = 0$. This transformation satisfies all the assumptions, mentioned above. Therefore, a matrix $A(t)$ can be reduced by (0.13) to the normal forms, given in Table 1.

In the case when $A(t)$ is invertible, its J-form allows us to find a formal solution to (0.11). This solution (found by Turrittin in [14], see also [6]) has the form

$$Y(t) = T(t)\Gamma(t)^{\Lambda} t^{M} \mathrm{e}^{Q(t)}, \tag{3.4}$$

where $T(t)$ is a root-meromorphic series with some $p \in \mathbb{N}$, $\Gamma(t)$ the Gamma function, $Q(t)$ a diagonal matrix, polynomial in $t^{1/p}$ of order not exceeding p, Λ a diagonal matrix with entries $k/p,\ k \in \mathbb{Z}$ and, M is a constant matrix in Jordan form, commuting with both Λ and $Q(t)$. Note that $-\deg Q(t) \leqslant 1$, where $r = 1$ is the Poincare rank of (0.13).

3.3. *Functional equations*

The transformation (0.14), generated by the functional equation (0.12), satisfies all the assumptions, mentioned above. Therefore, $A(t)$ can be reduced to its J-form by (0.14). Let

$$\phi(t) = t(1 + t^{-r}\gamma(t)), \tag{3.5}$$

where $r = m/p$ for some $m, p \in \mathbb{Z}^{+}$ and $\gamma(t) = \sum_{k=0}^{\infty} \gamma_k t^{-k/p}$ with $\gamma_k \in \mathbb{C}$ and $\gamma_0 \neq 0$. If $A(t)$ is invertible we can show that a formal solution to (0.12) has a similar to (3.4) form

$$Y(t) = T(t)g(t)^{\Lambda} t^{M} \mathrm{e}^{Q(t)}. \tag{3.6}$$

The differences between (3.4) and (3.6) are that the Γ-function is replaced by the function $g(t)$, satisfying the functional equation

$$g(\phi(t)) = tg(t) \tag{3.7}$$

and that the diagonal matrix $Q(t)$ is a polynomial in $t^{1/p}$ of order not exceeding mp. In the particular case $\phi(t) = t + 1$ solution (3.6) turns into (3.4).

Indeed, let the transformation (0.14) reduce $A(t)$ to its J-form, i.e. to a direct sum of the blocks

$$B(t) = t^{l/p}\left(\sum_{j=0}^{m} \lambda_j t^{-j/p} I + t^{-m/p} H\right),$$

where $l \in \mathbb{Z}$, $\lambda_j \in \mathbb{C}$ and H is the upper shift matrix (see Section 2). Moreover, $\lambda_0 \neq 0$ because $A(t)$ is invertible. The transformation

$$U(t) = Z(t)g^{l/p}(t) \tag{3.8}$$

reduces the corresponding functional equation $U(\phi(t)) = B(t)U(t)$ to

$$Z(\phi(t)) = \left(\sum_{j=0}^{m} \lambda_j t^{-j/p} I + t^{-m/p} H\right) Z(t). \tag{3.9}$$

The solution (3.6) is now a direct consequence of (3.8) and the following statement.

Statement 3.1. *Solution to* (3.9) *is given by*

$$Z(t) = X(t)\mathrm{e}^{q(t)}t^M, \tag{3.10}$$

where $q(t)$ is a polynomial in $t^{1/p}$ of degree not exceeding m, $X(t)$ is an invertible matrix holomorphic seried in $t^{-1/p}$ and M is a constant matrix in Jordan form. The matrices M and $X(t)$ commute with each other and with H.

Proof. The substitution $Z(t) = \mathrm{e}^{q(t)}t^M X(t)$ reduces (3.9) to

$$\mathrm{e}^{[q(\phi(t))-q(t)]}\left(\frac{\phi(t)}{t}\right)^M X(\phi(t)) = \sum_{j=0}^{m}\left(\lambda_j t^{-j/p} I + t^{-m/p} H\right) X(t). \tag{3.11}$$

We are looking for $q(t)$ and M, such that the coefficient of $X(\phi(t))$ in the left-hand side of (3.11) is equal to the coefficient of $X(t)$ in the right-hand side up to the terms of the order $\mathrm{O}(t^{-(m+1)/p})$. Then, taking into account (3.5), we get

$$\begin{aligned}&[q(\phi(t)) - q(t)]I + M\ln(1 + t^{-m/p}\gamma(t)) = \ln\lambda_0 I\\ &\quad+\ln\left(I + \lambda_0^{-1}\sum_{j=1}^{m}\lambda_j t^{-j/p} I + \lambda_0^{-1} t^{-m/p} H + \mathrm{O}(t^{-(m+1)/p})\right).\end{aligned} \tag{3.12}$$

Equating the leading coefficients of (3.12), we get $q_m = (p\ln\lambda_0)/(m\gamma_0)$ for the leading coefficient q_m of $q(t)$. The rest of the coefficients as well as the matrix M can be found recurrently. Moreover, the matrix $M = \mu I + \delta H$ with some $\mu, \delta \in \mathbb{C}$.

Now Eq. (3.11) can be written

$$X(\phi(t) = K(t)X(t), \tag{3.13}$$

where $K(t) = I + \mathrm{O}(t^{-(m+1)/p})$ and K commutes with H. According to Remark 2.1, we can find a root-holomorphic series $X(t)$, satisfying (3.13). The fact that $X(t)$ commutes with H follows then from system (2.11). □

3.4. On Eq. (3.7)

In general, the "special function" $g(t)$, defined by Eq. (3.7), requires some additional study. However, in some particular cases (3.7) could be reduced to a scalar difference equation (0.11). Indeed, let $v(t)$ satisfy the scalar equation

$$v(\phi(t)) = ev(t). \tag{3.14}$$

According to Statement 3.1, we can choose

$$v(t) = \mathrm{e}^{q(t)}t^M X(t), \tag{3.15}$$

where $q(t)$ is a polynomial in $t^{1/p}$ of degree m (note that the leading coefficient $q_m \neq 0$), M is a complex number and $X(t) = 1 + \mathrm{o}(1)$ is a holomorphic series in $t^{-1/p}$. Then

$$h(t) = \ln v(t) = q(t) + M \ln t + \ln X(t) \tag{3.16}$$

satisfies the equation

$$h(\phi(t)) = h(t) + 1. \tag{3.17}$$

Introducing now the new unknown function *f* by $f(h(t)) = g(t)$, we reduce (3.7) to the difference equation

$$f(h+1) = s(h)f(h), \tag{3.18}$$

where $t = s(h)$ is the inverse to $h = h(t)$, defined by (3.16). Under the assumption that $M = 0$ one can show that $s(h)$ is a root-meromorphic series in h (of degree $-p/m$), thus completing the reduction of (3.7) to the difference equation (3.18). □

Remark 3.1. In general, the condition $M = 0$ allows us to reduce the functional equation (0.12) with a root-meromorphic coefficient $A(t)$ to the difference equation (0.11), which also have a root-meromorphic coefficient.

Example 3.1. In the case $\phi(t) = \sqrt{t^2 + a^2}$ Eq. (3.17) has a solution $h(t) = t^2/a^2$. Then the difference equation $f(h+1) = \sqrt{h}\, f(h)$ is satisfied by $f(h) = a^{h/a^2}\Gamma^{1/2}\left(h/a^2\right)$, so that

$$g(t) = a^{t^2/a^2}\Gamma^{1/2}\left(\frac{t^2}{a^2}\right) \tag{3.19}$$

is a solution to (3.7).

Example 3.2. In the case $\phi(t) = t + \frac{1}{t}$ the function $h(t)$, defined by (3.17), is $h(t) = \frac{1}{2}t^2 - \frac{1}{2}\ln t + \mathrm{O}(t^{-2})$. As we see, $M = -\frac{1}{2}$ and so the coefficient $s(h)$ in the difference equation (3.18) is not a root-meromorphic series. In general, such an equation cannot be satisfied by (3.4). However, after a “small correction” $\phi(t) = t + \frac{1}{t} - \frac{1}{2}\frac{1}{t^3}$, the function h becomes $h(t) = \frac{1}{2}t^2 + \mathrm{O}(t^{-2})$. Then direct computations show that $s(h) = \sqrt{2h}(1 + \mathrm{O}(h^{-2}))$, so that $f(h) = 2^{1/2h}\Gamma^{1/2}(h)\tilde{x}(h)$, where $\tilde{x}(h) = 1 + \mathrm{O}(h^{-1})$ is a holomorphic series.

Finally, we get

$$g(t) = 2^{t^2/2}\Gamma^{1/2}(h(t))x(t), \tag{3.20}$$

where $x(t) = 1 + \mathrm{O}(t^{-1})$ is a holomorphic series.

4. Equations with a small parameter

4.1. Regularly perturbed equation (0.15)

Let $\mathscr{F}$ denote the field of root-meromorphic series in x^{-1}, i.e. $\mathscr{F} = \mathbb{C}\{x^{-1}\}$. According to Statement 1.3, we can assume that in (0.15)

$$A(x,\varepsilon)=\lambda(x,\varepsilon)I+\varepsilon^r D(x,\varepsilon), \tag{4.1}$$

where λ is a polynomial in $\varepsilon^{1/p}$ over $\mathscr{F}$ of order less than rp and $D(x,\varepsilon)$ is a matrix over $\mathscr{F}[[\varepsilon]]$. Let us assume at the beginning that the polynomial $\lambda=0$. Then (0.15) turns into the regularly perturbed equation

$$\partial_x Z(x,\varepsilon)=\left(\sum_{k=0}^{\infty}A_k(x)\varepsilon^{k/p}\right)Z(x,\varepsilon), \tag{4.2}$$

where A_k are matrices over $\mathscr{F}$. Without loss of generality, we put $p=1$.

We are looking for a series

$$Z(x,\varepsilon)=\sum_{k=0}^{\infty}Z_k(x)\varepsilon^k, \tag{4.3}$$

satisfying (4.2). Then the coefficients Z_k satisfy a reccurent system of nonhomogeneous linear differential equations of the form

$$\begin{aligned}&\partial_x Z_0-A_0Z_0=0,\\ &\partial_x Z_k-A_0Z_k=\sum_{j=1}^{k}A_{j,k}Z_{k-j},\quad k=1,2,\dots,\end{aligned} \tag{4.4}$$

where in general, $A_{j,k}$ are matrices over $\mathscr{F}$. In the particular case of Eq. (4.2), $A_{j,k}=A_j$ for all $k\in\mathbb{Z}^+$ and all $j=1,2,\dots,k$. According to (0.10), $Z_0(x)=T(x)x^N e^{Q(x)}$, where $T(x)$ is a root-meromorphic transformation, N is a nilpotent matrix in the Jordan form and

$$Q(x)=\operatorname{diag}(q_1(x),q_2(x),\dots,q_n(x)).$$

Note that $U=Tx^N$ is a matrix over the ring of polynomials $\mathscr{F}[\ln x]$.

Remark 4.1. In the case of a regular transformation (0.16) (i.e. $r=0$), we have $F=\mathrm{id}$ and $G=\partial_x$. Therefore, in the proof of Theorem 2.1 we have $F_{m+j}=T_{m+j}$ and $G_{m+j}=\partial_x T_{m+j}$ instead of (2.8). Let us also assume that the leading coefficient in the series (2.6) is T_0 instead of I. Then system (4.4) coincides with (2.10), where $Z_k=T_k$ and $B_k=0$ for all $k=0,1,2,\dots$.

Lemma 4.1. *Let the coefficients of the series* (4.3) *satisfy Eq.* (4.4), *where* A_0 *and* $A_{j,k}$, $k\in\mathbb{Z}^+$, $j=1,2,\dots,k$ *are matrices over* $\mathscr{F}$. *Then*

$$Z(x,\varepsilon)=X(x,\varepsilon)e^{Q(x)}, \tag{4.5}$$

where X is a matrix holomorphic series in ε *over the ring* $\mathscr{F}[\ln x]$.

Proof. We will use induction to show that $Z_k e^{-Q(x)}$ is a matrix over $\mathscr{F}[\ln x]$, $k=0,1,2,\dots$ For $k=0$ the statement follows from (0.10). A solution to the nonhomogeneous differential equation (4.4) with $k\geqslant 1$ is

$$Z_k(x) = Z_0(x) \int^x Z_0^{-1}(\tau) \sum_{j=1}^{k} A_{j,k}(\tau) Z_{k-j}(\tau)\,\mathrm{d}\tau.$$

According to the assumption of the induction,

$$Z_k(x)\mathrm{e}^{-Q(x)} = U(x)\mathrm{e}^{Q(x)} \int^x \mathrm{e}^{-Q(\tau)} R_k(\tau)\mathrm{e}^{Q(\tau)}\,\mathrm{d}\tau\,\mathrm{e}^{-Q(x)}, \tag{4.6}$$

where $R_k(x)$ is a matrix over $\mathscr{F}[\ln x]$. Since the matrix Q is diagonal, the statement of the lemma is a consequence of the following simple fact (see, for example [11]). □

Statement 4.1. *If $x\partial_x q(x)$ is a polynomial in $x^{1/p}$, $p \in \mathbb{Z}^+$, and if $r(x) \in \mathscr{F}[\ln x]$, then $\mathrm{e}^{q(x)} \int^x \mathrm{e}^{-q(\tau)} r(\tau)\,\mathrm{d}\tau \in \mathscr{F}[\ln x]$.*

4.2. Homomorphism δ

Let δ be the homomorphism of the ring $\mathscr{F}[\ln x]$, defined by

$$\delta \ln x = \ln x + 1 \quad \text{and} \quad \delta y = y \tag{4.7}$$

for all $y \in \mathscr{F}$. We continue δ on the ring of power series $(\mathscr{F}[\ln x])[[\varepsilon]]$ by $\delta\varepsilon = \varepsilon$. According to Lemma 4.1, the solution $Z(x,\varepsilon)$ is a matrix over the $(\mathscr{F}[\ln x])[[\varepsilon]]$-module M with generators $e_j = \mathrm{e}^{q_j(x)}$, where $q_j(x)$ are the entries of the diagonal matrix $Q(x)$. In fact, M is a differential module with differentiation ∂_x. It is easy to check that δ is a homomorphism of the differential module M (i.e. δ commutes with ∂_x) if we continue δ on M by $\delta e_j = e_j$.

By differentiating the identity $Z^{-1}Z = I$ we see that the inverse matrix Z^{-1} satisfies the equation

$$\partial_x Z^{-1} = -Z^{-1}A, \tag{4.8}$$

where $A(x,\varepsilon) = \sum_{k=0}^{\infty} A_k(x)\varepsilon^k$. On the other hand, applying δ to both sides of (4.2), we get

$$\partial_x(\delta Z) = A\delta Z. \tag{4.9}$$

Lemma 4.2. *The matrix $Z(x,\varepsilon)$ satisfies*

$$\delta Z(x,\varepsilon) = Z(x,\varepsilon)C(\varepsilon), \tag{4.10}$$

where $C(\varepsilon)$ is an invertible matrix over $\mathbb{C}[[\varepsilon]]$, commuting with $Q(x)$. The only eigenvalue of the leading coefficient C_0 of $C(\varepsilon)$ is 1.

Proof. Indeed, the combination of (4.8) and (4.9) yields $\partial_x(Z^{-1}\delta Z) = -Z^{-1}A\delta Z + Z^{-1}A\delta Z = 0$, which implies (4.10). Substituting (4.5) into (4.10), we get $X^{-1}(x,\varepsilon)\,\delta X(x,\varepsilon) = \mathrm{e}^{Q(x)}C(\varepsilon)\mathrm{e}^{-Q(x)}$. As the left-hand side of this equation is a matrix over

$(\mathscr{F}[\ln x])[[\varepsilon]]$ the matrix $C(\varepsilon)$ commutes with the diagonal matrix $\mathrm{e}^{Q(x)}$. Then C_0 commutes with $Q(x)$. Considering the leading order equation $\delta Z_0 = Z_0 C_0$ of (4.10) and using (0.10), we find out that $C_0 = x^{-N}\delta x^N = \mathrm{e}^{-N\ln x}\mathrm{e}^{N(\ln x+1)} = \mathrm{e}^N$. The proof is completed. □

For a matrix $C(\varepsilon)$ from Lemma 4.2, let us define

$$\ln C(\varepsilon) = -\sum_{k=1}^{\infty}\frac{1}{k}(I - C(\varepsilon))^k.$$

Simple algebra shows that $\ln C(\varepsilon)$ is a matrix over $\mathbb{C}[[\varepsilon]]$ with the leading term N and that $\mathrm{e}^{\ln C(\varepsilon)} = C(\varepsilon)$.

Lemma 4.3. *If a matrix $Z(x,\varepsilon)$ satisfies Lemmas* 1 *and* 2, *then*

$$Z(x,\varepsilon) = \bar{X}(x,\varepsilon)x^{N(\varepsilon)}\mathrm{e}^{Q(x)}, \tag{4.11}$$

where $\bar{X}$ is an invertible matrix over $\mathscr{F}[[\varepsilon]]$ and $N(\varepsilon) = \ln C(\varepsilon)$.

Proof. Based on Lemmas 4.1 and 4.2, we get $\delta X = \delta Z\mathrm{e}^{-Q} = ZC(\varepsilon)\mathrm{e}^{-Q} = X\mathrm{e}^{Q}C(\varepsilon)\mathrm{e}^{-Q} = XC(\varepsilon)$. Then $\delta[X(x,\varepsilon)x^{-\ln C(\varepsilon)}] = X(x,\varepsilon)C(\varepsilon)\exp(-[\ln x + 1]\ln C(\varepsilon)) = X(x,\varepsilon)x^{-\ln C(\varepsilon)}$. So, $\bar{X}(x,\varepsilon) = X(x,\varepsilon)\mathrm{e}^{-\ln C(\varepsilon)}$ is a matrix over $\mathscr{F}[[\varepsilon]]$ and the proof is completed. □

4.3. Solution to Eq. (0.15)

According to Lemma 4.3, Eq. (4.2) has a solution (4.11). Then the transformation (0.16), with $T(x,\varepsilon) = \bar{X}(x,\varepsilon)$, reduces Eq. (4.2) into $\partial_x Z = BZ$, where

$$B(x,\varepsilon) = \partial_x(x^{N(\varepsilon)}\mathrm{e}^{Q(x)})(x^{N(\varepsilon)}\mathrm{e}^{Q(x)})^{-1} = \partial_x Q(x) + x^{-1}N(\varepsilon).$$

The same transformation applied to Eq. (4.1) (i.e. without the assumption $\lambda = 0$) yields

$$B(x,\varepsilon) = \lambda(x,\varepsilon)I + \partial_x Q(x) + x^{-1}N(\varepsilon). \tag{4.12}$$

Let $H(\varepsilon) = \sum_{k=0}^{\infty} H_k\varepsilon^{k/p}$ be the J-form of the matrix $N(\varepsilon)$, where, for the sake of simplicity, p is taken the same as in (4.2). Note that H_0 is a nilpotent matrix that could differ from N, and that both $H(\varepsilon)$ and the reducing root-meromorphic transformation (a matrix over $\mathbb{C}\{\varepsilon\}$) commute with $Q(x)$. Thus, we get the following result.

Theorem 4.1. *A matrix A over the ring $\mathscr{F}[[\varepsilon]]$ could be reduced by a root-meromorphic transformation* (0.16) *to its J-form*

$$J(x,\varepsilon) = \Lambda(x,\varepsilon) + x^{-1}H, \tag{4.13}$$

where $\Lambda(x,\varepsilon) = \sum_{k=0}^{\infty}\Lambda_k(x)\varepsilon^{k/p}$ is a diagonal matrix series (*with some $p \in \mathbb{Z}^+$*) *over the field $\mathscr{F}$, $x\Lambda_{rp}$ is a polynomial in positive fractional powers of x, $x\Lambda_k$ are* *matrices over* $\mathbb{C}$ *for all* $k > rp$, H *a direct sum of upper shift matrices*, H *and* $\Lambda(x,\varepsilon)$ *commute.*

Let $\int \Lambda_k(x)\,\mathrm{d}x = Q_k(x) + P_k \ln x,\ k = 0, 1 \ldots, pr$, where Q_k and P_k are matrices over the fields $\mathscr{F}$ and $\mathbb{C}$, respectively. Solving Eq. (0.15), where $A(x,\varepsilon)$ has the form (4.13), we get the following corollary.

Corollary 4.1. *The solution to* (0.15) *is given by*

$$\begin{aligned} Y(x,\varepsilon) &= T(x,\varepsilon)x^{\sum_{j=0}^{\infty} P_j\varepsilon^{-r+j/p}+H} \exp\left(\sum_{j=0}^{pr} Q_j(x)\varepsilon^{-r+j/p}\right) \\ &= T(x,\varepsilon)x^{P(\varepsilon)+H}\mathrm{e}^{Q(x,\varepsilon)}, \end{aligned}$$

where $T(x,\varepsilon)$ *is a root-meromorphic transformation* (*over* $\mathscr{F}$) *and* $P_j = \Lambda_j$ *for* $j = pr+1, pr+2, \ldots$

4.4. *Solution to Eq. (0.17) with an invertible matrix* $A(x,\varepsilon)$

According to Statement 1.3, we can assume that the coefficient $A(x,\varepsilon)$ in (0.17) has the form (4.1) with $r = 1$. Moreover, let us assume at the beginning that the polynomial $\lambda = 1$. Then (0.17) can be written as

$$Z(x+\varepsilon,\varepsilon) = \left(I + \varepsilon\sum_{k=0}^{\infty} D_k(x)\varepsilon^{k/p}\right) Z(x,\varepsilon), \tag{4.14}$$

where D_k are matrices over $\mathscr{F}$. Without loss of generality, we put $p = 1$.

As in Section 4.1, we are looking for a series (4.3) satisfying (4.14). According to (1.3), direct substitution of (4.3) into (4.14) yields

$$\begin{aligned} \partial_x Z_0 &= D_0 Z_0, \\ \partial_x Z_k &= D_0 Z_k + \sum_{j=1}^{k}\left(D_j Z_{k-j} - \frac{\partial_x^{j+1} Z_{k-j}}{(j+1)!}\right), \quad k = 1, 2, \ldots \end{aligned} \tag{4.15}$$

Statement 4.2. *System* (4.15) *has the form* (4.4), *where* $A_{j,k}$ *are matrices over* $\mathscr{F}$ *for all* $k \in \mathbb{Z}^+$ *and* $j = 1, 2 \ldots, k$.

Proof. It is sufficient to show that the higher derivatives $\partial_x^l Z_k$ also satisfy equations of the form (4.4), i.e. that there exist matrices $A_{j,k,l}$ over $\mathscr{F}$ satisfying

$$\partial_x^l Z_k = \sum_{j=0}^{k} A_{j,k,l} Z_{k-j} \tag{4.16}$$

for any $l \in \mathbb{Z}^+$ and any $k \in \mathbb{N}$.

We prove (4.16) by induction. Indeed, differentiating the first Eq. (4.15), we get $\partial_x^l Z_0 = A_{0,0,l} Z_0$, where $A_{0,0,l}$ are polynomials in D_0 and derivatives of D_0. Thus, $A_{0,0,l}$ are matrices over $\mathscr{F}$.

Suppose now that (4.16) holds for all $k = 0, 1, 2 \ldots, m-1$, where $m \in \mathbb{Z}^+$, and for all $l \in \mathbb{Z}^+$. Then, according to (4.15) and (4.16), $\partial_x Z_m = \sum_{j=0}^m A_{j,m,1} Z_{m-j}$. Differentiating this equation, we get

$$\partial_x^2 Z_m = \sum_{j=0}^{m} (\partial_x A_{j,m,1} Z_{m-j} + A_{j,m,1} \partial_x Z_{m-j}) = \sum_{j=0}^{m} A_{j,m,2} Z_{m-j},$$

where

$$A_{j,m,2} = \partial_x A_{j,m,1} + \sum_{n=0}^{j} A_{j-n,m,1} A_{n,m-j+n,1}$$

are matrices over $\mathscr{F}$. Differentiating the equation for $\partial_x^2 Z_m$, we can prove (4.16) for $k = m$ and any $l = 3, 4, \ldots$ We can now use the induction arguments to complete the proof. □

An immediate consequence of Statement 4.2 and Lemma 4.1 is that the solution $Z(x, \varepsilon)$ to (4.14) can be written in the form (4.5), where the matrix $Q(x)$ is defined by the solution (0.10) to $\partial_x Z_0 = D_0 Z_0$.

Our next observation is that $\tau = \mathrm{e}^{\varepsilon \partial_x}$ is a homomorphism of the differential module M, defined in Section 4.2, where we put $\tau e_j = \mathrm{e}^{\tau q_j(x)}$. Eq. (4.14) can be rewritten now as $\tau Z = EZ$, where E denotes the matrix coefficient in the right-hand side of (4.14). Moreover, τ commutes with δ since ∂_x commutes with δ. Then δZ is another solution to (4.14). Applying τ to the identity $Z^{-1} Z = I$, we get $\tau(Z^{-1}) = Z^{-1} E^{-1}$ provided that E is invertible. So, $\tau(Z^{-1} \delta Z) = Z^{-1} E^{-1} E \delta Z = Z^{-1} \delta Z$. Since the fixed points of the homomorphism τ are elements of $\mathbb{C}[[\varepsilon]]$, we conclude that

$$\delta Z(x, \varepsilon) = Z(x, \varepsilon) C(\varepsilon),$$

where $C(\varepsilon)$ is the matrix over $\mathbb{C}[[\varepsilon]]$. Using the arguments of Lemma 4.2, we can show that $C(\varepsilon)$ commutes with $Q(x)$ and the leading term C_0 of $C(\varepsilon)$ has the only eigenvalue 1. Now solution $Z(x, \varepsilon)$ satisfies all the assumptions of Lemma 4.3, so that we get the following lemma.

Lemma 4.4. *Eq.* (4.14) *has a solution of the form* (4.11).

Let us now replace our previous assumption $\lambda(x, \varepsilon) = 1$ in (4.1) by a less restrictive one: $\lambda(x, \varepsilon) = \sum_{j=0}^{p-1} \lambda_j(x) \mathrm{e}^{j/p}$, where $\lambda_0(x) \neq 0$. Let $g(x, \varepsilon)$ denote a solution to the scalar equation

$$g(x + \varepsilon, \varepsilon) = \lambda(x, \varepsilon) g(x, \varepsilon). \tag{4.17}$$

Then direct computations show that the transformation

$$Y(x,\varepsilon)=g(x,\varepsilon)Z(x,\varepsilon) \tag{4.18}$$

reduces Eq. (0.17) and (4.1) to the form (4.14).

Suppose now that $\lambda(x,\varepsilon)=\varepsilon^{j/p}(\lambda_j(x)+\mathrm{O}(\varepsilon^{-1/p}))$, where $\lambda_j\neq 0$ and $j>0$. Then the transformation

$$Y(x,\varepsilon)=\varepsilon^{jx/p\varepsilon}Z(x,\varepsilon), \tag{4.19}$$

following by the corresponding root-meromorphic transformation (0.18), reduces the problem to the previous case $\lambda_0\neq 0$.

Since $\deg\det T^{-1}=-\deg\det T$ for any root-meromorphic transformation T, it is easy to see that $\deg\det[T^{-1}(x+\varepsilon,\varepsilon)A(x,\varepsilon)T(x,\varepsilon)]=\deg\det A(x,\varepsilon)$. Consider now the case when in (4.1) $\lambda(x,\varepsilon)=0$ (recall that for difference equations $r=1$). Then $\deg\det D(x,\varepsilon)=\deg\det A(x,\varepsilon)-1$.

Let us reduce $D(x,\varepsilon)$ to the form (4.1) by the corresponding root-meromorphic transformation. In this process we either split the system, or obtain a new polynomial $\tilde\lambda(x,\varepsilon)$ and a new matrix $\tilde D(x,\varepsilon)$. If $\tilde\lambda(x,\varepsilon)\neq 0$, we can use transformations (4.18) and (4.19) to reduce the system to the case $\tilde\lambda(x,\varepsilon)=1$. Otherwise, we have $\deg\det\tilde D(x,\varepsilon)=\deg\det A(x,\varepsilon)-2$ and continue the process. As at each step $\deg\det\tilde D(x,\varepsilon)$ cannot be negative, we can continue this process only finitely many times. Thus, a finite number of transformations (4.18) and (4.19) and of root-meromorphic transformations reduce any Eqs. (0.17) and (4.1) to the case $\lambda(x,\varepsilon)=1$.

In order to construct a solution to (0.17) we need to solve Eq. (4.17).

Statement 4.3. *Solution to the scalar equation* (4.17) *with* $\lambda_0\neq 0$ *is given by*

$$g(x,\varepsilon)=t(x,\varepsilon)x^{\rho x/\varepsilon+p(\varepsilon)}\mathrm{e}^{q(x,\varepsilon)}, \tag{4.20}$$

where $t(x,\varepsilon)$ *is a root-holomorphic series over* $\mathscr{F}$, $\rho\in\mathbb{C}$, $p(\varepsilon)$ *and* $q(x,\varepsilon)$ *are polynomials in* $\varepsilon^{-1/p}$ *of orders not exceeding p over the fields* $\mathbb{C}$ *and* $\mathscr{F}$, *respectively.*

Proof. The substitution

$$g(x,\varepsilon)=\mathrm{e}^{\gamma(x,\varepsilon)}z(x,\varepsilon),$$

where $\gamma(x,\varepsilon)=\sum_{j=0}^{p-1}\gamma_j(x)\varepsilon^{(j-p)/p}$, reduce (4.17) to

$$\exp\left(\sum_{k=1}^{\infty}\frac{\varepsilon^k\partial_x^k}{k!}\gamma(x,\varepsilon)\right)z(x+\varepsilon,\varepsilon)=\lambda(x,\varepsilon)z(x,\varepsilon). \tag{4.21}$$

The exponential in the left-hand side can be written as $\exp\big(\sum_{j=0}^{p-1}\varepsilon^{j/p}\partial_x\gamma_j(x)+\mathrm{O}(\varepsilon)\big)$. So, it is a holomorphic series in $\varepsilon^{1/p}$. If we equate the first p coefficients of this series with the corresponding coefficients of $\lambda(x,\varepsilon)$, we then reduce (4.17) to a (scalar) equation of the form (4.14). Equating the corresponding coefficients, we get

$$\mathrm{e}^{\partial_x\gamma_0(x)}=\lambda_0(x),\quad \partial_x\gamma_1(x)=\frac{\lambda_1(x)}{\lambda_0(x)},\ldots, \tag{4.22}$$

i.e. each $\partial_x\gamma_j$ is a rational function of $\lambda_0, \dots, \lambda_{p-1}$. Thus, every $\gamma_j = q_j + c_j \ln x$, $j = 1, 2, \dots, p-1$, where $c_j \in \mathbb{C}$ and $q_j \in \mathscr{F}$. As it follows from (4.22), $\gamma_0 - \rho x \ln x = q_0(x) + c_0 \ln x$, where $\rho = -\deg \lambda_0$, $c_0 \in \mathbb{C}$ and $q_0 \in \mathscr{F}$. So,

$$\gamma(x,\varepsilon) - \rho\frac{x}{\varepsilon}\ln x = \sum_{j=0}^{p-1}(q_j(x) + c_j \ln x)\varepsilon^{(j-p)/p}. \tag{4.23}$$

Applying Lemma 4.4 to the scalar equation (4.14) for $z(x,\varepsilon)$, we obtain $z(x,\varepsilon) = t(x,\varepsilon)\exp[q_p(x) + c_p \ln x]$, where $c_p \in \mathbb{C}$, $q_p \in \mathscr{F}$ and $t(x,\varepsilon)$ is a root holomorphic series over $\mathscr{F}$. To complete the proof, it remains to define $q(x,\varepsilon) = \sum_{j=0}^{p} q_j(x)\,\varepsilon^{(j-p)/p}$ and $p(\varepsilon) = \sum_{j=0}^{p} c_j \varepsilon^{(j-p)/p}$. $\square$

Combining Lemma 4.4 with (4.18)–(4.20), we obtain the following "discrete analog" of Corollary 4.1.

Theorem 4.2. *The solution to* (0.17) *with an invertible matrix* $A(x,\varepsilon)$ *is given by*

$$Y(x,\varepsilon) = T(x,\varepsilon)(\varepsilon^S x^R)^{x/\varepsilon} \mathrm{e}^{Q(x,\varepsilon)} x^{P(\varepsilon)+H}, \tag{4.24}$$

where $T(x,\varepsilon)$ is a root-meromorphic transformation (over $\mathscr{F}$), S and R are diagonal matrices of rational numbers, $Q(x,\varepsilon) = \sum_{j=0}^{p} Q_j(x)\varepsilon^{(j-p)/p}$, where all Q_j are diagonal matrices over $\mathscr{F}$ and Q_p is a polynomial in positive fractional powers of x, $P(\varepsilon) = \sum_{j=0}^{\infty} P_j \varepsilon^{(j-p)/p}$, where all P_j are diagonal matrices over $\mathbb{C}$, H is a direct sum of the upper shift matrices that commutes with the diagonal matrices S, R, Q and P.

4.5. J-form

In order to obtain J-form of a matrix $A(x,\varepsilon)$ with respect to near-similarity transformations (0.18), we need the following fact.

Statement 4.4. *Solution to the $n \times n$ equation*

$$Y(x+\varepsilon,\varepsilon) = \left(I + \frac{\varepsilon}{x}H\right)Y(x,\varepsilon), \tag{4.25}$$

where H is an $n \times n$ upper shift matrix, is given by

$$Y(x,\varepsilon) = U(x,\varepsilon)x^H, \tag{4.26}$$

where U is a holomorphic series in ε/x over $\mathbb{C}$ and U commutes with H.

Proof. The substitution $Y = x^H Z$ reduces (4.25) to

$$Z(x+\varepsilon,\varepsilon) = \left(1 + \frac{\varepsilon}{x}\right)^{-H}\left(I + \frac{\varepsilon}{x}H\right)Z(x,\varepsilon).$$

The coefficient of this system has the form $I + \sum_{k=2}^{\infty} \tilde{f}_k(H)\,(\varepsilon/x)^k$, where $\tilde{f}_k$ are some polynomials. Looking for a solution $Z = I + \sum_{k=1}^{\infty} Z_k \varepsilon^k$, we get

$$\partial_x Z_k = \sum_{j=1}^{k} \left[\frac{\tilde{f}_{j+1}(H) Z_{k-j}}{x^{k+1}} - \frac{\partial_x^{j+1} Z_{k-j}}{(j+1)!} \right], \tag{4.27}$$

where $Z_0 = I$. We can now use induction to show that (4.27) implies $Z_k(x) = x^{-k} f_k(H)$, $k \in \mathbb{Z}^+$, where f_k is a polynomial. The proof is completed. $\square$

Corollary 4.2. *A matrix A over the ring* $\mathscr{F}[[\varepsilon]]$ *could be reduced by a root-meromorphic transformation* (0.18) *to its J-form*

$$J(x, \varepsilon) = \varepsilon^S \left[\Lambda(x, \varepsilon) + \frac{\varepsilon}{x} H \right], \tag{4.28}$$

where $\Lambda(x, \varepsilon) = \sum_{k=0}^{\infty} \Lambda_k(x) \varepsilon^{k/p}$ *is a diagonal holomorphic matrix series over the field* $\mathscr{F}$, *matrices S and H are the same as in Theorem* 4.2, *H commutes with* $\Lambda(x, \varepsilon)$ *and S.*

Proof. Let $T(x, \varepsilon) G(x, \varepsilon)$ denote the right-hand side of (4.24). The transformation $Y(x, \varepsilon) = T(x, \varepsilon) Z(x, \varepsilon)$ reduces the coefficient $A(x, \varepsilon)$ in Eq. (0.17) to

$$B(x, \varepsilon) = G(x + \varepsilon, \varepsilon) G^{-1}(x, \varepsilon) = \varepsilon^S \tilde{\Lambda}(x, \varepsilon) \left(1 + \frac{\varepsilon}{x}\right)^H. \tag{4.29}$$

Direct computation shows that

$$\begin{aligned} \tilde{\Lambda}(x, \varepsilon) &= \left(1 + \frac{\varepsilon}{x}\right)^{x/\varepsilon R + P(\varepsilon)} (x + \varepsilon)^R \, \mathrm{e}^{Q(x+\varepsilon, \varepsilon) - Q(x, \varepsilon)} \\ &= x^R \exp\left\{ \left[\left(\frac{x}{\varepsilon} + 1 \right) R + P(\varepsilon) \right] \ln\left(1 + \frac{\varepsilon}{x}\right) + Q(x + \varepsilon, \varepsilon) - Q(x, \varepsilon) \right\}. \end{aligned}$$

According to Theorem 4.2, the exponent in the latter expression is a diagonal matrix, which is represented by a holomorphic series in $\varepsilon^{1/p}$ over $\mathscr{F}$. Therefore, $\tilde{\Lambda}(x, \varepsilon) = \mathrm{diag}(\tilde{\lambda}_1(x, \varepsilon), \ldots, \tilde{\lambda}_n(x, \varepsilon))$ is a matrix over $\mathscr{F}[[\varepsilon^{1/p}]]$.

As a consequence of Statement 4.4, we get

$$U(x + \varepsilon, \varepsilon)(x + \varepsilon)^H = \left(I + \frac{\varepsilon}{x} H\right) U(x, \varepsilon) x^H. \tag{4.30}$$

For every Jordan block H_i of dimension n_i in H we can find the corresponding matrix U_i using Statement 4.4. Note that U, the direct sum of matrices U_i, commutes with Λ and S. Applying to B the transformation (0.18) with $T = U^{-1}$ and using (4.30), we get

$$\begin{aligned} U(x + \varepsilon, \varepsilon) B(x, \varepsilon) U^{-1}(x, \varepsilon) &= \varepsilon^S \tilde{\Lambda}(x, \varepsilon) U(x + \varepsilon, \varepsilon) \left(1 + \frac{\varepsilon}{x}\right)^H U^{-1}(x, \varepsilon) \\ &= \varepsilon^S \tilde{\Lambda}(x, \varepsilon) \left(1 + \frac{\varepsilon}{x} H\right). \end{aligned} \tag{4.31}$$

Finally, we apply the root-meromorphic transformation (0.18) with the shearing matrix

$$V(x,\varepsilon) = \operatorname{diag}\left(1, \tilde{\lambda}_1^{-1}(x,\varepsilon), \tilde{\lambda}_1^{-1}(x+\varepsilon,\varepsilon)\tilde{\lambda}_2^{-1}(x,\varepsilon), \ldots, \right.$$
$$\left. \prod_{k=1}^{n_i-1} \tilde{\lambda}_k^{-1}(x+(n_i-k-1)\varepsilon,\varepsilon)\right)$$

for each block H_i. This transformation reduces (4.31) to (4.28), where $\Lambda(x,\varepsilon) = \operatorname{diag}(\lambda_1(x,\varepsilon), \ldots, \lambda_n(x,\varepsilon))$ and

$$\lambda_j(x,\varepsilon) = \tilde{\lambda}_j(x,\varepsilon)\prod_{k=1}^{j-1}\frac{\tilde{\lambda}_{j-k}(x+k\varepsilon,\varepsilon)}{\tilde{\lambda}_{j-k}(x+(k-1)\varepsilon,\varepsilon)}, \quad j=1,2,\ldots,n_i-1. \qquad \square$$

Corollary 4.3. *We have established that both transformation* (0.16) *and* (0.18) *satisfy Assumption T* (*with* $F = \mathrm{id}$, $G = \varepsilon^{-r}\partial_x$ *and* $F = \mathrm{e}^{\varepsilon\partial_x}$, $G = 0$, *respectively*). *Thus Table* 1 *describes four normal forms of the near-similarity transformatios* (0.16) *and* (0.18).

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